★ Coordinate Changes for Integrals

Remark: In Calc I, solved integrals like ① $\int_{x=0}^{3} x e^{x^2} dx$ $u = x^2$, $du = 2x\,dx$ ← parameter change (reparametrization of $\mathbb{R}$), differential changes too

① In double integrals we made the polar coordinate change...
$dA_{cart} = r\, dA_{polar}$

Q. how to do this more generally?

A. We'll use a Jacobian

Def^n: Suppose
$$\begin{cases} x_1 = x_1(u_1, u_2, \dots, u_n) \\ x_2 = x_2(u_1, u_2, \dots, u_n) \\ x_3 = x_n(u_1, u_2, \dots, u_n) \end{cases}$$
a coordinate change by diff. functions, The (signed) Jacobian change of the coordinate is
$$\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)} = \det\begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix}$$

Ex. Comp the signed Jacobian of polar transform: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Sol. $\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \cos\theta\, r\cos\theta - \sin\theta(-r\sin\theta) = r(\sin^2\theta + \cos^2\theta) = r$

NB: Swapping order of (r,θ) to (θ, r)
$$\frac{\partial(x,y)}{\partial(\theta,r)} = \det\begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{bmatrix} = \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix} = -r\sin^2\theta - r\cos^2\theta = -r$$

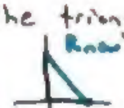
The unsigned Jacobian is $\left|\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}\right|$

Prop: If $f(x_1, x_2, \dots, x_n)$ is a cnts function & $\begin{cases} x_1 = x_1(u_1, u_2, \dots, u_n) \\ x_2 = x_2(u_1, u_2, \dots, u_n) \\ x_n = x_n(u_1, u_2, \dots, u_n) \end{cases}$ is a diff. coordinate transform
$$\int\!\!\int_{R_{old}} dV_{old} = \int_{R_{new}} f(x_1(u_1, u_2, \dots, u_n), \dots, x_n(u_1, \dots, u_n)) \left|\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}\right| dA_{new}$$

Ex. Comp $\iint_R (x - 3y)\, dA$ for R, the triangle w/ vertices $(0,0)$ $(1,2)$ $(2,1)$

Sol 1. (Comp. by hand)

Sol 2. (using transformation)



R_{new} $\begin{cases} x = 2\alpha + 1\beta \\ y = 1\alpha + 2\beta \end{cases}$ b/c $(\alpha, \beta) = (1,0)$ we have $(x(\alpha,\beta), y(\alpha,\beta)) = (2,1)$

$(\alpha,\beta) = (0,1)$ $(x(\alpha,\beta), y(\alpha,\beta)) = (1,2)$

By its geometry, this linear change takes R_{new} to R_{old}. $(\alpha,\beta) = (0,0)$ $(x(\alpha,\beta), y(\alpha,\beta)) = (0,0)$

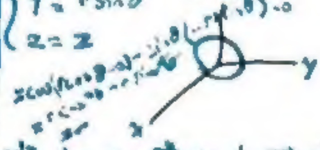
$R_{new} = \{(\alpha,\beta) \mid 0 \le \alpha \le 1,\ 0 \le \beta \le 1 - \alpha\}$

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{bmatrix} = \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$\therefore \iint_R (x - 3y)\, dA_{old} = \iint_{R_{new}} (2\alpha + \beta - 3(\alpha + 2\beta)) \left|\frac{\partial(x,y)}{\partial(\alpha,\beta)}\right| dA_{new}$$
$$= \int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha} (-\alpha - 5\beta)3\, d\beta\, d\alpha$$
$$= \int_{\alpha=0}^{1} -3\left[(1-\alpha)(\alpha + \tfrac{5}{2}(1-\alpha))\right] d\alpha$$
$$= -\tfrac{3}{2}\int_{\alpha=0}^{1} (5 - 8\alpha + 3\alpha^2)\, d\alpha$$
$$= -\tfrac{3}{2}\left[5\alpha - 4\alpha^2 + \alpha^3\right]_{\alpha=0}^{1} = -\tfrac{3}{2}[5 - 4 + 1] = -3$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Generalizing polar coordinates in 3-Space

I. Naive way: Cylindrical coords: Just parametrize 1 plane by polar coords, leave orthogonal axis alone

$\therefore$ when we comp integrals in cylindrical coords, we need to multiply the diff by r

↳ True of all cylindrical changes

Ex. Comp $\iiint_R (x + y + z)\, dV$ for R the solid in first octant & below $4 - x^2 - y^2 = z$

Sol. in cylindrical coords $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ $R_{cyl} = \{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2,\ 0 \le z \le 4 - r^2\}$

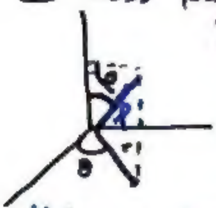
$$\therefore \iiint_{R_{cart}} (x + y + z)\, dV_{old} = \iiint_{R_{cyl}} (r\cos\theta + r\sin\theta + z)\, r\, dV_{new}$$
$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2}\int_{\theta=0}^{\pi/2} (r^2\cos\theta + r^2\sin\theta + zr)\, d\theta\, dz\, dr$$
$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2} (2r^2 + z\tfrac{\pi}{2})\, dz\, dr$$

$$\int_{\theta=0}^{\pi/2} (r^2\cos\theta + r^2\sin\theta + zr)$$
$$= r\left[r\sin\theta - r\cos\theta + z\theta\right]_{\theta=0}^{\pi/2}$$
$$= r\left[r + \tfrac{z\pi}{2} - (-r)\right]$$

$$\int_{z=0}^{4-r^2} (2r^2 + z\tfrac{\pi}{2})\, dz$$
$$= \left[2r^2 z + \tfrac{\pi}{4}z^2 r\right]_{z=0}^{4-r^2}$$
$$= (8r^2 - 2r^4 + \tfrac{\pi}{4}(16 - 8r^2 + r^4))$$

$$= \int_{r=0}^{2} (8r^2 - 2r^4 + \tfrac{\pi}{4}(16r - 8r^3 + r^5))\, dr$$
$$= \left[\tfrac{8}{3}r^3 - \tfrac{2}{5}r^5 + \tfrac{\pi}{4}(8r^2 - 2r^4 + \tfrac{1}{6}r^6)\right]_0^2$$
$$= \frac{64}{3} - \frac{64}{5} + \frac{\pi}{4}(32 - 32 + \tfrac{32}{3}) - 0$$
$$= \frac{64}{3} - \frac{64}{5} + \frac{8\pi}{3}$$

II Less Naive way: Spherical Coords



In spherical coords, we paramatize points (x,y,z) using 3 pieces of data:

- ρ = distance from origin
- θ = angle made w/ pos. x-axis and point $(x,y,0)$
- φ = angle made w/ pos. z-axis & pt. (x,y,z)

Note $\sin(\varphi) = \frac{r}{\rho}$, so $r = \rho\sin(\varphi)$

in our paramatization

$$\begin{cases} x = r\cos\theta = \rho\sin(\varphi)\cos(\theta) \\ y = r\sin\theta = \rho\sin(\varphi)\sin(\theta) \\ z = \rho\cos(\varphi) \end{cases}$$

$$dA_{cart} = \rho^2\sin(\varphi)\,dA_{sph}$$